Lecture 17

MTH 161

Lecture 17

Curve Sketching

So far we have done some aspects in Chapter 1 like domain, range, symmetry, limits, continuity, Derivatives and tangents in Chapter 2, extreme values, intervals of increase / decrease, concavity, inflection points in chapter 3.

Put all this information together.

GUIDELINES FOR SKETCHING A CURVE

A. Domain

B. Intercepts

C. Symmetry (Even / odd, periodic)

D. Asymptotes

E. Intervals of Increase / Decrease

F. Local Max / Min

G. Concavity / Inflection Points.

H. Sketch the curve.

Example 1

Use the guidelines to sketch the curve $y = \frac{2x^2}{x^2 - 1}$

A. Domain is $x^2 - 1 \neq 0 \Rightarrow x^2 \neq 1 \Rightarrow x \neq \pm 1$

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

B. Intercepts

Y-intercept is when $x = 0$, $y = \frac{2 \cdot 0}{-1} = 0$

X-intercept is when $y = 0$, $0 = \frac{2x^2}{x^2 - 1} \Rightarrow 2x^2 = 0 \Rightarrow x = 0$

So both x and y intercept are $(0,0)$.

c) Symmetry $f(x) = f(-x)$

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} = f(x)$$

So function is even.

D) Asymptotes

V.A : $x^2 - 1 = 0 \Rightarrow x = \pm 1$

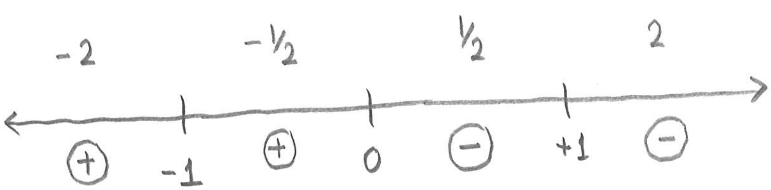
HA $\lim\limits_{x \to \pm\infty} \dfrac{2x^2}{x^2-1} = \lim\limits_{x \to \pm\infty} \dfrac{2}{1 - \frac{1}{x^2}} = 2$

E) Intervals of Increase / Decrease

$$f'(x) = \frac{(x^2-1)4x - 2x^2(2x)}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

So critical numbers are $x = 0, x = -1, x = +1$



So function decreasing on $(0,1) \cup (1,\infty)$

Increasing on $(-\infty,-1) \cup (-1,0)$

F) Local Max / Min

By first Derivative Test, $f(0) = 0$ is the local max.

G) Concavity / Inflection points

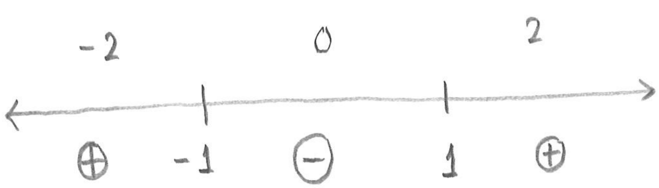
$$f''(x) = \left(\frac{-4x}{(x^2-1)^2} \right)' = \frac{(x^2-1)^2(-4) + 2(x^2-1) \cdot 2x\,(-4x)}{(x^2-1)^4}$$

$$= \frac{\cancel{(x^2-1)}\left[-4(x^2-1) + 16x^2\right]}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{12x^2+4}{(x^2-1)^3}$$

So $f''(x) = 0 \Rightarrow 12x^2 + 4 = 0$ cannot happen

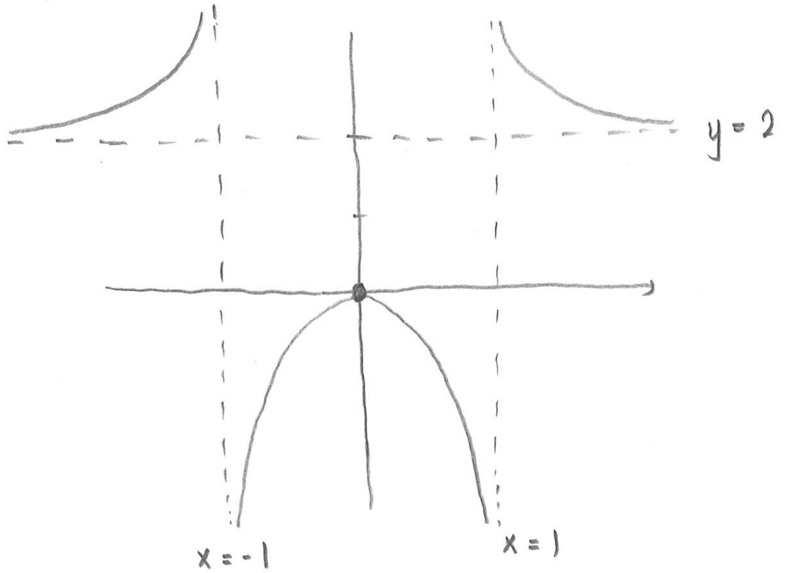
$f''(x)$ is not defined when $x = \pm 1$



So CU on $(-\infty, -1) \cup (1, \infty)$

CD on $(-1, 1)$

However $1, -1$ are not I.P because ± 1 are not in Domain of f.



Ex Sketch the graph of $f(x) = \dfrac{\cos x}{2+\sin x}$.

A. Domain $\quad 2 + \sin x = 0 \quad \Rightarrow \sin x = -2$

Since, $-1 \leq \sin x \leq 1$, $\sin x \neq -2$. Hence Denominator is never 0.

Hence, Domain is $(-\infty, \infty)$.

B. Intercepts

Y-intercept is when $x = 0$, $y = \dfrac{\cos 0}{2+\sin 0} = \dfrac{1}{2}$

X-intercept is when $y = 0$, $\dfrac{\cos x}{2+\sin x} = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots = \dfrac{\pi}{2} + 2K\pi, \dfrac{3\pi}{2} + 2K\pi$

c) Symmetry

$$f(-x) = \frac{\cos(-x)}{2+\sin(-x)} = \frac{\cos x}{2-\sin x}$$

$$-f(x) = \frac{-\cos x}{2+\sin x}$$

Hence f is neither even nor odd.

However, $f(x+2\pi) = f(x)$, and so f is periodic and has period 2π. This means that we only need to worry about when $0 \leq x \leq 2\pi$ and the complete the curve by translation.

D) Asymptotes

NO V.A., NO H.A. $\left(\text{since } \lim_{x \to \pm\infty} \cos x \text{ DNE and } \lim_{x \to \pm\infty} \sin x \text{ DNE} \right)$

E) Intervals of Increase / Decrease

$$f'(x) = \frac{(2+\sin x)\cdot(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

So what are critical points ?

Denominator is never 0, so critical points are when $f'(x) = 0$

$$-2\sin x - 1 = 0 \Rightarrow 2\sin x = -1 \Rightarrow \sin x = -\frac{1}{2}$$

$$\Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

π $\quad$ $\frac{3\pi}{2}$

0 $\ominus$ $\frac{7\pi}{6}$ $\oplus$ $\frac{11\pi}{6}$ $\ominus$ 2π

Increasing on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

Decreasing on $\left(0, \frac{7\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

F) <u>Local Max / Min</u>

By first derivative Test, local max $f\left(\frac{7\pi}{6}\right) = \dfrac{\cos\left(\frac{7\pi}{6}\right)}{2+\sin\left(\frac{7\pi}{6}\right)} = \dfrac{-\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \dfrac{-\sqrt{3}}{3}$

Local min $f\left(\frac{11\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$.

$$f''(x) = \left(\frac{-2\sin x - 1}{(2+\sin x)^2} \right)'$$

$$= \frac{(2+\sin x)^2 \cdot (-2\cos x) - (-2\sin x - 1)\, 2(2+\sin x) \cdot \cos x}{(2+\sin x)^4}$$

$$= \frac{\cos x\,(2+\sin x)\left[-2(2+\sin x) + 4\sin x + 2\right]}{(2+\sin x)^4}$$

$$= \frac{\cos x\,(-4 - 2\sin x + 4\sin x + 2)}{(2+\sin x)^3}$$

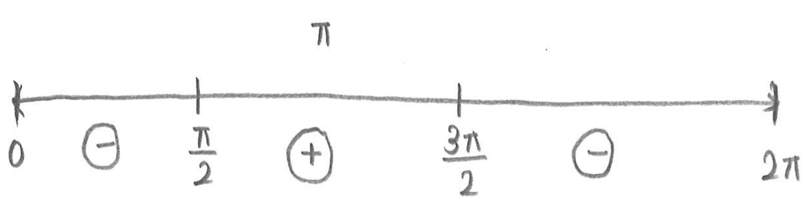
$$= \frac{2\cos x\,(\sin x - 1)}{(2+\sin x)^3}$$

So when is $f''(x) = 0$ or $f''(x)$ not defined ?

Since denominator is never 0 , $f''(x)$ is always defined .

$f''(x) = 0 \Rightarrow 2\cos x(\sin x - 1) = 0 \Rightarrow \cos x = 0$ or $\sin x = 1$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$



So C.U on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

CD on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

Then, I.P are

$\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$